The Alpha Invariant and K-stability of $\mathbb{Q}$-Fano Varieties.

Recall: Let X be a cpct n-dim'l smooth Fano Variety, and $G \subseteq Aut(X)$ (possibly trivial) ~~a subgroup~~. Let ω be a fixed G invariant Kähler form in $C_1(X)$. set

$$P_G(X) = \left\{ \varphi \in C^2_{\mathbb{R}}(X) \;\middle|\; \varphi \text{ is } G\text{-invariant}, \sup \varphi = 0, \text{ and } \omega + \frac{i}{2\pi}\partial\bar\partial\varphi > 0 \right\}$$

Then

$$\alpha_G(X) = \sup\left\{ \alpha > 0 \;\middle|\; \exists\, C(\alpha) \text{ s.t. } \int_X e^{-\alpha\varphi}\omega^n < C(\alpha)\ \forall\ \varphi \in P_G(X,\omega) \right\}.$$

$\alpha_G(X)$ is independent of ω. Then Tian showed if $\alpha_G(X) > \frac{n}{n+1}$, then X admits a KE metric.

Thm: (Donaldson) if (X,L) has cscK metric $\omega \in C_1(L)$, then (X,L) is K-stable.

Thus: $\alpha_G(X) > \frac{n}{n+1} \Rightarrow (X, K_X)$ is K-stable.

Question: Can we prove this directly?

Def'n: A test configuration for (X,L) is a scheme $\mathcal{X}$, and an ample line bundle $\mathcal{L} \to \mathcal{X}$, together with a $\mathbb{C}^*$ equivariant map $\pi : (\mathcal{L} \to \mathcal{X}) \to \mathbb{C}$, with π flat, such that the generic fibre $\mathcal{L}_t \to \mathcal{X}_t \cong L \to X$.

$\pi^{-1}(0)$ inherits a $\mathbb{C}^*$ action, thus we can compute the total weight of the induced action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$

Let $w(k)$ be this weight: then $w(k) = b_0 k^{n+1} + b_1 k^n + \cdots$

Then $DF(\mathcal{X},\mathcal{L}) = \dfrac{a_0 b_1 - a_1 b_0}{a_0^2}$ where $H^0(X,L) = a_0 k^n + a_1 k^{n-1} + \cdots$ are the R.R. coefficients. [(X,L) k-stable if $\forall$ test configurations $(\mathcal{X},\mathcal{L})$ we have $DF(\mathcal{X},\mathcal{L}) \geq 0$]

Lemma 1: $H^0(\mathcal{X}_0, \mathcal{L}_0^k) \cong \pi_* \mathcal{L}^k \big|_{\{0\}} \cong H^0_{\mathcal{X}}(\mathcal{L}^k) \big/ t H^0_{\mathcal{X}}(\mathcal{L}^k)$

pf $0 \to t H^0_{\mathcal{X}}(\mathcal{L}^k) \longrightarrow H^0_{\mathcal{X}}(\mathcal{L}^k) \xrightarrow{|_{t=0}} H^0_{\mathcal{X}}(\mathcal{X}_0, \mathcal{L}_0^k)$ gives the inclusion $H^0_{\mathcal{X}}(\mathcal{X},\mathcal{L}^k) \big/ t H^0(\mathcal{X},\mathcal{L}^k) \hookrightarrow H^0(\mathcal{X}_0, \mathcal{L}_0^k)$.

The other inclusion follows from flatness and non-sense. By Theorem 12.11 in chapter 3 of Hartshorne, any $s \in \pi_* \mathcal{L}\big|_{\{0\}}$ extents to a meromorphic section of $\pi_* \mathcal{L}$ over $\mathbb{C}$, which is holomorphic near zero. Choose a polynomial

$p(t)$ s.t. $\begin{cases} p(0)=1 \\ p(t)\cdot s \text{ is holomorphic} \end{cases}$. Then $[p.s] \in H^0(\mathcal{X},\mathcal{L}) / tH^0(\mathcal{X},\mathcal{L})$ is the desired section. ▨

This will be useful later.

K-stability is a hard condition to check. We need some "simple" family of test configs.

Deformation to the Normal Cone.

Let $Z \subseteq X$ be a subscheme (think subvariety). Define a sheme $\mathcal{X} \xrightarrow{\pi} \mathbb{C}$ by $\mathcal{X} = Bl_{Z\times\{0\}} X \times \mathbb{C}$.

~~If~~ $\mathcal{X} \xrightarrow{\tilde{\pi}} X \times \mathbb{C} \xrightarrow{P_2} \mathbb{C}$, then $\pi = P_2 \circ \tilde{\pi}$. Let E be the exceptional divisor of $\tilde{\pi}$, and set $\mathcal{L}_c = \tilde{\pi}^* L \otimes \mathcal{O}(-cE)$. ~~we~~ for c small $\mathcal{L}_c$ is ample. The $\mathbb{C}^*$ action on $\mathcal{X}$ is induced from the trivial action on $X \times \mathbb{C}$ (acting only on $\mathbb{C}$). This action is trivial on $\mathcal{X}_0 \setminus E$, but <u>not</u> trivial on E.

For this family we can do many computations explicitly. see, e.g. Ross-Thomas.

__Remark__: Blowing up $X \times \mathbb{C}$ in $Z \times \{0\}$ is the scheme theoretic blow up in the Ideal $I_{Z\times\{0\}} = I_Z + (t)$. This agrees w/ the usual blow up when Z is a variety.

We can do this more generally. Choose a subvariety Z_1 then $\mathcal{X}_1 = Bl_{Z_1 \times\{0\}} X \times \mathbb{C}$. Let Z_2 be any $\mathbb{C}^*$ invariant subscheme of $\pi_1^{-1}(0) = (\mathcal{X}_1)_0 \subseteq \mathcal{X}_1$ and take $\mathcal{X}_2 = Bl_{Z_2}\mathcal{X}_1$ __etc__. This is the same as scheme theoretically blowing up ~~the~~ a __flag variety__

__Def'n__: let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{N-1} \subseteq \mathcal{O}_X$ be a sequence of coherent ideals. Then set

$$J = I_0 + tI_1 + \cdots + I_{N-1}t^{N-1} + (t^N).$$

Then we set $\mathcal{B} := Bl_J X \times \mathbb{C} \xrightarrow{\pi} \mathbb{C}$, and let E be the exceptional divisor. Let $L \to X$ be ample ~~...~~ $X \times \mathbb{C} \xrightarrow{p_1} X$, and $\mathcal{L} = p_1^* L$. then $\mathcal{L}(-E)$ is (semi)ample (replace $L \to L^r$ if necessery) and this gives a (semi) test configuration.

Theorem 2: (Odaka).
A polarized Variety (X,L) is K-stable if and only if $DF((\mathcal{X},\mathcal{L})) > 0$ for all (semi) test configurations obtained from Flag Ideals as before.

Theorem 3: Let (X,L), B, J be as before. Then we have

$$DF\left((Bl_J(X\times\mathbb{C}),\ \mathcal{L}(-E))\right) =$$

$$\frac{1}{2\,n!\,(n+1)!}\Big\{-n(L^{n-1}\cdot K_X)(\mathcal{L}(-E))^{n+1} + (n+1)L^n(\mathcal{L}(-E))^n\cdot\pi^*p_1^*K_X$$

$$+ (n+1)(L^n)(\mathcal{L}(-E))^n\cdot K_{B/X\times\mathbb{C}}\Big\}$$

where intersection numbers are computed on X or $\overline{B} = Bl_J(X\times\mathbb{P}^1)$.

Recall:

$$\mathrm{Sesh}(J;(X,L)) := \sup\{c>0 \mid \pi^*L(-cE) \text{ is ample}\}.$$

Idea: Relate Log Canonical Threshold to Seshadri Constant which we use to bound terms appearing in the DF invariant.

pf of Theorem 3: we want to use.

Lemma (Mumford). Let V be a vectorspace over $\mathbb{C}$, with a $GL(n,\mathbb{C})$ action on $V \otimes_{\mathbb{C}} \mathbb{C}[t]$ where the action on V is trivial, and the weight on t is (-1). Let

$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{N-1} \subseteq V_N = V = \cdots = V$, set $\mathcal{V} := \sum t^i V_i$

a sub $\mathbb{C}[t]$ module of $V \otimes_{\mathbb{C}} \mathbb{C}[t]$. Then, the total weight on $\mathcal{V}/t\mathcal{V}$ is equal to $-\dim\left((V \otimes_{\mathbb{C}} \mathbb{C}[t])/\mathcal{V}\right)$.

We want to use this on $\mathcal{V} = H^0(\mathfrak{X}, \mathcal{L}^k(-E))$ by Lemma 1.

Now $H^0(\mathfrak{X}, \mathcal{L}^k(-E)) = H^0(X \times \mathbb{C}, \pi_* \mathcal{L}^k \otimes J^k)$

Since sections of $\mathcal{L}^k(-E)$ correspond to sections of $\mathcal{L}^k$ on the base ~~[illegible]~~ vanishing to order k on "J".

Now $J^k = I_0^k \oplus I_0^{k-1} I_1 t \oplus (I_0^{k-2} I_1^2 + I_0^{k-1} I_2) t^2 \oplus \cdots \oplus (t^{Nk})$

Now since $I_0 \subseteq I_1 \subseteq \cdots$ we get

$$\mathcal{L}^k \otimes J^k = \bigoplus_{j=0}^{Nk} \tilde{I}_j t^j \otimes \mathcal{L}^k \quad \text{and} \quad \tilde{I}_j \subseteq \tilde{I}_{j+1} \subseteq \cdots \subseteq \mathcal{O}_X.$$

So $H^0(X \times \mathbb{C}, \mathcal{L}^k \otimes J^k) = \bigoplus_{j=0}^{Nk} t^j H^0(X, \mathcal{L}^k \otimes \tilde{I}_j)$

Thus, setting $V = H^0(X, \mathcal{L}^k)$, we see

$$w(k) = -\dim\left(H^0(X,\mathcal{L}^k) \otimes_{\mathbb{C}} \mathbb{C}[t] \Big/ H^0(X\times\mathbb{C}, \mathcal{L}^k \otimes J^k) \right)$$
$$= -\dim\left(H^0(X\times\mathbb{C}, \mathcal{L}^k) \Big/ H^0(X\times\mathbb{C}, \mathcal{L}^k\otimes J^k) \right)$$

We now use:

$$0 \to \mathcal{L}^k \otimes J^k \to \mathcal{L}^k \to \mathcal{L}^k \otimes \mathcal{O}/J^k \to 0$$

The Long exact sequence in cohomology gives

$$0 \to H^0(X\times\mathbb{C}, \mathcal{L}^k \otimes J^k) \to H^0(X\times\mathbb{C}, \mathcal{L}^k) \to H^0(X\times\mathbb{C}, \mathcal{L}^k / J^k\mathcal{L}^k)$$
$$\to H^1(\mathcal{L}^k J^k) \to \cdots$$

So $\dim H^0(X\times\mathbb{C}, \mathcal{L}^k) \Big/ H^0(\mathcal{L}^k J^k) = \dim H^0\left(\mathcal{L}^k / J^k\mathcal{L}^k\right) - \dim H^1(\mathcal{L}^k J^k)$

Lemma: $h^i(\mathcal{L}^k J^k) = O(k^{n-1}) \ \forall\, i>0.$

So $w(k) = -\dim H^0(X\times\mathbb{C}, \mathcal{L}^k\otimes J^k) + \dim H^0(X\times\mathbb{C}, \mathcal{L}^k) + O(k^{n-1})$

Now apply Riemann-Roch

we want to apply this formula when $L = -rK_X$ where $r \in \mathbb{Z}_{>0}$ is large so that $\mathcal{L} = \pi^* p_1^* (\mathcal{O}_X(-rK_X))$ satisfies $\mathcal{L}(-E)$ is ample. Then

$$DF(B, \mathcal{L}(-E)) \cdot 2n!(n+1)! = -((\bar{\mathcal{L}}-E)^n \cdot \bar{\mathcal{L}}) + ((\bar{\mathcal{L}}-E)^n \cdot ((n+1)rK_{B/X\times\mathbb{C}} - nE))$$

Proposition: $-((\bar{\mathcal{L}}-E)^n \cdot \bar{\mathcal{L}}) \geq 0$ for any flag ideal J.

So it suffices to prove the 2^{nd} term is positive.

Corollary A: if $\exists\, \varepsilon \geq 0$ s.t. $\boxed{\left(\frac{n+1}{n}\right) K_{B/X\times\mathbb{C}} - \mathrm{Sesh}(J, (X\times\mathbb{C}, -K_{X\times\mathbb{C}}))\, E > \varepsilon E}$

then $DF(B, \mathcal{L}(-E)) \geq 0$

pf need to compute.

$$rn(\bar{\mathcal{L}}-E)^n \cdot \left(\frac{(n+1)}{n} K_{B/X\times\mathbb{C}} - \frac{1}{r}E\right)$$

Note that since $\mathcal{L}(-E)$ is ample, $\mathcal{L} = \pi^* p_1^*(-K_X^r)$ we know that $\frac{1}{r} < \mathrm{Sesh}(J, (X\times\mathbb{C}, -K_{X\times\mathbb{C}}))$.

Now $((\bar{\mathcal{L}}-E)^n \cdot E) > 0$ by another result of Odaka.

So $(\bar{\mathcal{L}}-E)^n \cdot \left(\frac{n+1}{n} K_{B/X\times\mathbb{C}} - \frac{1}{r}E\right) > (\bar{\mathcal{L}}-E)^n \cdot \varepsilon E \geq 0$. ∎

<u>NTS</u>: Condition in Corollary A holds if $\alpha(X) > \frac{n}{n+1}$.
To do this, relate α to the log canonical threshold.

<u>Log Canonical Threshold</u>:

Let $a \subseteq \mathcal{O}_X$ be a non-zero coherent sheaf of ideals (we will often assume $a = I_D$ for some divisor D), and $\pi : \hat{X} \to X$ a log resolution of singularities. Let $\sum_i E_i$ be a divisor w/ simple normal crossings (snc) s.t.

- $D = \sum a_i E_i$, where $a \cdot \mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{X}}(-D)$
- $K_{\hat{X}/X} = \sum k_i E_i$, <u>then</u> $\mathrm{lct}(a) = \min_i \frac{1+k_i}{a_i}$

<u>Recall</u>: $\pi : \hat{X} \to X$ is a log resolution if D, ~~has normal~~ and $K_{\hat{X}/X}$ have normal crossings.

<u>Def'n</u>: $\mathrm{lct}(X) = \inf_{m \in \mathbb{Z}_{>0}} \inf_D \mathrm{lct}(X, \frac{1}{m} D)$ where 2^{nd} infimum runs over effective divisors which are linearly equivalent to $-mK_X$. ($\mathrm{lct}(X,D) = \mathrm{lct}(I_D)$.)

if $a = (f_1, \dots, f_k)$, then $\mathrm{lct}(a) = \sup\left\{ s \,\middle|\, \frac{1}{\left(\sum_{i=1}^{k} |f_i|^2\right)^s} \in L^1_{loc}(X) \right\}$

__Def'n:__ If $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$ are ideals, then we define the __mixed__ log canonical threshold

$$\mathrm{lct}_{(X,\mathfrak{b})}(\mathfrak{a}) := \sup\{c \geq 0 \mid (X, \mathfrak{b}\cdot\mathfrak{a}^c) \text{ is l.c.}\}.$$

__re:__ if $\mathfrak{b} = (f)$, $\mathfrak{a} = (g)$ then

$$\mathrm{lct}_{(X,\mathfrak{b})}(\mathfrak{a}) = \sup\left\{ s \;\Big|\; \frac{1}{|f|^2 |g|^{2s}} \in L^1_{loc}(X) \right\}.$$

__Note:__ in general ~~[illegible]~~ $\mathrm{lct}_{(X,\mathfrak{b})}(\mathfrak{a})$ may be negative. However, we shall always deal with $(\mathfrak{a},\mathfrak{b})$ s.t. $\mathrm{lct}_{(X,\mathfrak{b})}(\mathfrak{a}) \geq 0$.

__Example:__ Consider $I = (y^3 - x^2) \subseteq \mathbb{C}[x,y]$. Define $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ by $(u,v) \xrightarrow{\pi} (u^3 v, u^2 v)$. Then

$$\pi^* I = u^6 v^2 (v-1), \quad \pi^* K_X = u^4 v \, du \wedge dv.$$

__So__ $\mathrm{lct}(I) = \frac{5}{6}$.

__Theorem__ (Demailly) $\alpha(X) = \mathrm{lct}(X)$ (and $\alpha_G(X) = \mathrm{lct}_G(X)$).

the "G-invariant versions"

Proposition: Let X be a $\mathbb{Q}$-Fano variety. If $\mathrm{lct}(X) > 0$ then we have:

$$\mathrm{Sesh}(J, (X \times \mathbb{C}, -K_{X\times\mathbb{C}})) \leq \frac{1}{\mathrm{lct}(X)} \min_i \left\{ \frac{1 + a_i - b_i}{c_i} \right\} \qquad (*)$$

where $B = Bl_J X \times \mathbb{C}$, $K_{B/X\times\mathbb{C}} = \sum_i a_i E_i$

$b_i := \mathrm{ord}_{E_i}(X \times \{0\})$ and $c_i = \mathrm{ord}_{E_i}(J)$.

pf

Fact (Ross-Thomas)

$$\mathrm{Sesh}(J, (X \times \mathbb{C}, -K_{X\times\mathbb{C}})) = \min_j (\mathrm{Sesh}(I_j, (X, -K_X)))$$

Thus: suffices to show $(*)$ for I_0 instead of J.

Let $c \in \mathbb{Q}_{>0}$ satisfy $c < \mathrm{Sesh}(I_0, (X, -K_X))$.

Let D be a divisor corresponding to $\{s=0\}$ for $s \in H^0(X, I_0^{mc} \mathcal{O}_X(-mK_X))$ (for $m \gg 0$, a global section exists by ampleness). Let $b = I_{\{X \times \{0\}\}} \subseteq X \times \mathbb{C}$.

Then:

$$\mathrm{lct}(X) \overset{\text{def'n}}{\leq} \mathrm{lct}(X, I_D) \overset{(*)}{\leq} \mathrm{lct}_{(X\times\mathbb{C}, b)} I_{D\times\mathbb{C}}$$

(*) follows from inversion of adjunction or equivalently the L^2 extension Theorem of Oshawa-Takegoshi

$$\leq \mathrm{lct}_{(X\times\mathbb{C}, b)} I_0^c \quad \text{since } I_0^c \geq I_{D\times\mathbb{C}} \text{ by def'n of } D.$$

<u>But</u> $I_0^c \leq J^c$ since $J = I_0 + tI_1 + \cdots$

<u>So</u> $\mathrm{lct}_{(X\times\mathbb{C}, b)} \leq \mathrm{lct}_{(X\times\mathbb{C}, b)} J^c = \frac{1}{c} \mathrm{lct}_{(X\times\mathbb{C}, b)} J$

$$\leq \frac{1}{c} \min_i \left\{ \frac{1 + a_i - b_i}{c_i} \right\}.$$

This last inequality follows from the fact that

$\pi : B \to X\times\mathbb{C}$ is a birational map. The generators of J (say $\{g_1, \ldots, g_k\}$) ~~are not~~ have

$$b \cdot \left(\sum_{i=1}^{k} |g_i|^2 \right)^s \notin L^1_{loc}(X\times\mathbb{C}) \quad \text{if} \quad s \geq \left\{ \frac{1 + a_i - b_i}{c_i} \right\} \text{ for any } i$$

<u>But</u> this estimate is not sharp, unless π is a log resolution. (just check in local coordinates!)

it follows that $c \leq \frac{1}{\mathrm{lct}(X)} \min_i \left\{ \frac{1+a_i-b_i}{c_i} \right\}$

as desired.

<u>Now</u> if $\mathrm{lct}(X) > \frac{n}{n+1}$, <u>then</u>

$$\left(\frac{n+1}{n}\right) K_{B/X\times\mathbb{C}} - \mathrm{sesh}(\mathcal{J}, (X\times\mathbb{C}, -K_{X\times\mathbb{C}}))\, E$$

$$> \frac{n+1}{n} K_{B/X\times\mathbb{C}} - \frac{n+1}{n} \min_j \left\{ \frac{1+a_j-b_j}{c_j} \right\} \sum c_i E_i$$

<u>But</u> $K_{B/X\times\mathbb{C}} = \sum a_i E_i$, so

$$\cancel{= \frac{n+1}{n} \sum \left[a_i - (1+a_i-b_i) \right] E_i}$$

$$\frac{n+1}{n} \sum_i \left[a_i - c_i \min_j \left\{ \frac{1+a_j-b_j}{c_j} \right\} \right] E_i \geqslant 0.$$

$$\| $$

$$\frac{n+1}{n} \sum_i \left\{ \left(\frac{1+a_i-b_i}{c_i} - \min_j \left\{ \frac{1+a_j-b_j}{c_j} \right\} \right) + \frac{b_i-1}{c_i} \right\} c_i E_i$$

<u>Note</u>: $\frac{b_i-1}{c_i} \geqslant 0$ since $b_i \in \mathbb{Z}_{>0}$ (b_i correspond to the zeroes of $\pi^* t$ which is hol'c.)